Lie-Cartan pairs

D. KASTLER

Dep. de Physique de Luminy
Université d'Aix Marseille II
and CPT-CNRS, Luming, France

R. STORA

LAPP, Annecy - France
Division Théorique
CERN - Genève (Switzerland)

Abstract. *The algorithms common to exterior derivation, exterior covariant derivation and vector valued cohomology of Lie-Algebras are presented within a unified frame.*

We present a unified description of the following algorithms: exterior derivative on a manifold, covariant derivative on smooth vector bundles, and cohomology of Lie algebras (over arbitrary commutative algebras) with values in representation spaces. The striking similarity of the formulae defining the exterior derivative, covariant exterior derivative and coboundary operator of Lie algebra cohomology thus appears as the fact that they are all special cases of a construction pertaining to pairs (L, A) of a Lie algebra L and a commutative algebra A with interrelations («*Lie-Cartan pairs*»). The archetype of this is the pair $L = \mathscr{X}(M), A = C^\infty(M), M$ a manifold, $\mathscr{X}(M)$ the Lie algebra of smooth vector fields on M. L is represented (in this case injectively) in the derivations of A, and is at the same time an A--module. One has furthermore the relations:

Key Words: Lie Cartan pairs; Cohomology of Lie algebras; Anomalies in Gauge Theory.
1980 Mathematics Subject Classification: 17 B 40; 17 B 56; 81 E 10.

$$(a\xi)\,b = a(\xi b), \qquad a, b \in A, \xi \in L,$$

$$[\xi, a\eta] = a[\xi, \eta] + (\xi a)\eta, \qquad \xi, \eta \in L, a \in A.$$

Axiomatizing these properties with a general - non necessarily injective - Lie--algebra representation: $L \to \mathrm{Der}\, A$ allows to encompass the following situation occuring in Lie algebra cohomology: L a Lie algebra (and hence a module) over a commutative algebra A, with trivial action of L on A: $\xi a = 0$ for all $\xi \in L$, $a \in A$.

For a general Lie-Cartan pair (L, A), and a module V over A, we define a *connection* ρ as a linear map of L into the linear ($= \mathbb{R}$ – or $\mathbb{C}$ – linear) operators of V, behaving as follows under multiplication by elements of A:

$$\rho(\xi)\,(aX) = a\rho(\xi)X + (\xi a)X, \qquad \xi \in L, a \in A, X \in V.$$

The corresponding *curvature*:

$$\Omega(\xi, \eta) = [\rho(\xi), \rho(\eta)] - \rho([\xi, \eta])$$

is then, as usual, an endomorphism of the module V. The «connection» ρ is called *local* whenever it commutes with the module-multiplication of L: $\rho(a\xi) = a\rho(\xi)$, $a \in A$, $\xi \in L$. Given a connection ρ, one constructs the usual analytical apparatus on $\wedge^*(L, V)$ (on $\wedge^*_A(L, V)$ if ρ is local): a «*derivative*» δ_ρ and a «*generalized Lie derivative*» $\Theta_\rho(\xi)$, $\xi \in L$. These objects, and the interior product $i(\xi)$, $\xi \in L$, then fulfill the usual relations cf. (51) to (55) below. In particular, the curvature is the square of the derivative:

$$\delta^2_\rho = \Omega \wedge$$

and one has the Cartan relation

$$\delta_\rho \,.\, i(\xi) + i(\xi)\,.\,\delta_\rho = \Theta(\xi).$$

This holds in the case of general representation spaces V, for which $\wedge^*(L, V)$ and $\wedge^*_A(L, V)$ are not algebras. The proof of these facts obtained by identifying derivations of a Grassmann algebra by the way in which they act on the elements of grade zero and one, hence does not reveal their true nature, which is more general, since they hold in the absence of a wedge product.

This paper is in fact intended as a future reference for a study of the algebric structure of anomalies of gauge fields.

[1] DEFINITIONS. A *Lie-Cartan pair* (L, A) is a couple of a real (complex) Lie algebra L, and a unital commutative real (complex) algebra A, equipped with

bilinear products [1]

(1) $(\xi, a) \in L \times A \to \xi a \in A$

and

(2) $(a, \xi) \in A \times L \to a\xi \in L$

such that:

(i) the product (1) defines a homomorphism of the Lie algebra L into the Lie algebra Der A of derivations of A:

(3) $\xi(ab) = (\xi a)\, b + a(\xi b), \qquad \xi \in L, a, b \in A$

(4) $[\xi, \eta]a = \xi(\eta a) - \eta(\xi a), \qquad \xi, \eta \in L, a \in A$

(ii) the product (2) makes L a unital A-module:

(5) $a(b\xi) = (ab)\xi, \qquad a, b \in A, \xi \in L$

(6) $1\xi = \xi, \qquad \xi \in L$

(iii) we have in addition the properties [2]

(7) $(a\xi)b = a(\xi b), \qquad a, b \in A, \xi \in L$

(8) $[\xi, a\eta] = a[\xi, \eta] + (\xi a)\eta, \quad a \in A, \xi, \eta \in L.$

Given a Lie-Cartan pair (L, A) and a left A-module V (i.e. a representation vector space of A), a *V-connection* [3] is an assignement ρ, to each $\xi \in L$, of a map: $\rho(\xi) : V \to V$ such that

(9)
$$\begin{cases} \rho(\xi)(X + Y) = \rho(\xi)X + \rho(\xi)Y \\ \rho(\xi)(aX) = a\rho(\xi)X + (\xi a)X \end{cases}, \qquad \begin{cases} \xi \in L, a \in A. \\ X, Y \in V, \end{cases}$$

assignement linear in the sense

(10) $\rho(\alpha\xi + \beta\eta) = \alpha\rho(\xi) + \beta\rho(\eta), \qquad \xi, \eta \in L \cdot \alpha, \beta \in R \ (\in \mathbb{C})$

The V-connection ρ is called *local* [4] whenever one has

---

[1] We say that (L, A) is a *real*, resp. a *complex* Lie-Cartan pair if L and A are over the real, resp. the complex numbers.

[2] Observe that (4) implies $[a\xi, \eta] = a[\xi, \eta] - (\eta a)\xi, a \in A, \xi, \eta \in L$.

[3] Note that the notion of V-connection depends on the specification of both Lie-Cartan pair (L, A) and the left A-module V.

[4] If we want to stress the role of the algebra A, we shall say *A-local* instead of local.

(11) $$\rho(a\xi) = a\rho(\xi), \qquad a \in A, \quad \xi \in L$$

and *flat (or a representation of L on V)* whenerer

(12) $$\rho([\xi, \eta]) = \rho(\xi)\,\rho(\eta) - \rho(\eta)\,\rho(\xi), \qquad \xi, \eta \in L.$$

The coherence of the definitions (10), (11) and (12) follows from (i) in

[2] LEMMA. *Let (L, A) be Lie-Cartan pair, with V a left A-module*

(i) *Let $\xi \to \rho(\xi)$ be an assignment, to the $\xi \in L$, of maps: $V \to V$ fulfilling (9). Then the maps*

(13) $$a\rho(\xi) + b\rho(\eta) - \rho(a\xi + b\eta), \qquad a, b \in A, \xi, \eta \in L$$

and

(14) $$\Omega(\xi, \eta) = \rho(\xi)\,\rho(\eta) - \rho(\eta)\,\rho(\xi) - \rho([\xi, \eta]), \qquad \xi, \eta \in L$$

are endomorphisms (A-linear maps) of V. For ρ a V-connection, Ω is called the curvature of ρ, *with $\Omega(\xi, \eta)$ the* value of the curvature for $\xi, \eta \in L$

(ii) *The set of V-connections is a convex set.*

Proof. The operator (13) is obviously additive. We have, for $X \in V$, a, b, $u \in A$

(15) $$\begin{aligned} \{a\rho(\xi) + (b\rho(\eta)\}(uX) &= a\rho(\xi)(uX) + b\rho(\eta)(uX) \\ &= a[u\rho(\xi) + (\xi u)X] + b\,[u\rho(\eta) + (\eta u)X] \\ &= u\,[a\rho(\xi) + b\rho(\eta)]X + [(a\xi + b\eta)u]X \end{aligned}$$

hence

(16) $$\begin{aligned} &\{a\rho(\xi) + b\rho(\eta) - \rho(a\xi + b\eta)\}(uX) \\ &\qquad = u\,[a\rho(\xi) + b\rho(\eta) - \rho(a\xi + b\eta)]X \end{aligned}$$

$\Omega(\xi, \eta)$ is also obviously additive. We have, for $X \in V$, $u \in A$

(17) $$\begin{aligned} \rho(\xi)\rho(\eta)(uX) &= \rho(\xi)[u\rho(\eta)X + (\eta u)X] \\ &= u\rho(\xi)\rho(\eta)X + (\xi u)\rho(\eta)X + (\eta u)\rho(\xi)X + (\xi\eta u)X \end{aligned}$$

hence

(18) $$\Omega(\xi, \eta)(uX) = u\Omega(\xi, \eta)X.$$

(ii) Let $\alpha', \alpha'' \in R (\in C)$ be such that $\alpha' + \alpha'' = 1$. For ρ', ρ'' two V-connections, $\rho = \alpha'\rho' + \alpha''\rho''$ obviously fulfills (10) and the first property (9). Moreover one has, for $a \in A, X \in V, \xi \in L$

$$\begin{aligned}
\rho(\xi)(aX) &= \alpha'\rho'(\xi)(aX) + \alpha''\rho''(\xi)(aX) \\
&= \alpha' a\rho'(\xi)X + \alpha'(\xi a)X + \alpha'' a\rho''(\xi)X + \alpha''(\xi a)X \qquad (19)\\
&= a\rho(\xi)X + (\xi a)X.
\end{aligned}$$

[3] LEMMA. *Let (L, A) be a Lie-Cartan pair. A* subpair *of (L, A) is a couple (L', A') of a Lie subalgebra L' of L and a subalgebra A' of A such that one has $\xi a \in A'$ and $a\xi \in L'$ for all $\xi \in L'$ and $a \in A'$. The subpair (L, A) is itself a Lie--Cartan pair for the restriction of the products (1) and (2).*

Given a real (complex) Lie-Cartan pair (L, A) the subpair $(L, \mathbb{R})$ $((L, \mathbb{C}))$ is called the depletion of (L, A)).

Proof. Obvious.

The next Lemma describes two opposite extremes of Lie-Cartan pairs yielding two classes of examples

[4] LEMMA. *Let (L, A) be a Lie-Cartan pair. It is called* injective *whenever*

$$\xi a = 0 \quad \textit{for all } a \quad \Rightarrow \quad \xi = 0 \qquad (20)$$

and degenerate *whenever* (5)

$$\xi a = 0 \quad \textit{for all} \quad \xi \in L \quad \textit{and} \quad a \in A. \qquad (21)$$

Then

(i) *all the injective Lie-Cartan pairs (and only those) are obtained* (6) *as the subpairs (L, A) of the Lie-Cartan pair* (Der A, A) *where A is any unital commutative real (complex) algebra, with* Der A *the Lie algebra of derivations of A, and*

$$(a\xi)b = a(\xi b), \quad a, b \in A, \ \xi \in \operatorname{Der} A \qquad (22)$$

(ii) *all the degenerate Lie-Cartan pairs (and only those) are obtained* (5) *as the pairs (L, A) of a unital commutative real (complex) algebra A, and Lie algebra L over A* (7), *with*

---

(5) Note that the depletion of any Lie-Cartan pair (L, A) is degenerate, since Der $\mathbb{R} =$ $=$ Der $\mathbb{C} = 0$.

(6) Bijectively.

(7) I.e. L is a real (complex) Lie Algebra which is a unital left A-module for which the Lie bracket is A-bilinear.

$$(23) \qquad \xi a = 0, \qquad \xi \in L, a \in A.$$

Let further (L, A) be a Lie-Cartan pair: if it is degenerate, each V-connection yields A-linear maps $\rho(\xi)$ for all $\xi \in L$. Conversely, the existence of a V-connection with this property for an A-separating A-module V implies that (L, A) is degenerate.

Proof. Let A be a unital commutative real (complex) algebra. For $\xi \in \mathrm{Der}\, A$ and $a \in A$, $a\xi$ as defined by (22) also belongs to $\mathrm{Der}\, A$, indeed (8)

$$(24) \qquad \begin{aligned}(a\xi)(bc) = a\{\xi(bc)\} &= a\{(\xi b)c + b(\xi c)\}\\ &= a(\xi b)c + ba(\xi c)\\ &= \{(a\xi)b\}c + b\{(a\xi)c\}.\end{aligned}$$

In this way $\mathrm{Der}\, A$ obviously becomes a left A-module, and $(\mathrm{Der}\, A, A)$ a Lie--Cartan pair: indeed, [1] (i) is automatic, [1] (ii) follows from (22) which coincides with (7), and we have, for $\xi, \eta \in L, a, b \in A$

$$(25) \qquad \begin{aligned}[\xi, a\eta]b = \xi\{(a\eta)b\} - (a\eta)(\xi b) &= \xi\{a(\eta b)\} - a\{\eta(\xi b)\}\\ &= a\{\xi(\eta b)\} + (\xi a)(\eta b) - a\{\eta(\xi b)\}\\ &= a\{[\xi, \eta]b\} + (\xi a)(\eta b)\\ &= \{a[\xi, \eta] + (\xi a)\eta\}b.\end{aligned}$$

Any subpair (L, A) of $(\mathrm{Der}\, A, A)$ then obviously fulfills (20) since $L \in \mathrm{Der}\, A$. Conversely, (20) means that the representation of L in $\mathrm{Der}\, A$ is faithful, and hence L can be considered as a Lie sub-algebra, and sub left A-module, of $\mathrm{Der}\, A$.

(ii) Let A be a unital commutative real (complex) algebra, with L a Lie algebra over A, we thus have a bilinear product (2) satisfying [1] (ii). The definition (23) then defines a bilinear product (1) trivially fulfilling [1] (i) and (7). Moreover (8) holds by the A-bilinearity of the Lie bracket, since $\xi a = 0$. Conversely, any degenerate Lie-Cartan pair (L, A) fulfills (8) with $\xi a = 0$, hence the Lie bracket of L is A-bilinear (right A-linearity implies left A-linearity owing to antisymmetry of the Lie bracket), therefore arises from a Lie algebra L over A.

Let (L, A) be a Lie-Cartan pair: if (L, A) is degenerate any V-connection ρ, V a left A-module, is A-linear owing to the second line in (9) with $\xi a = 0$. Conversely let V be a left A-module which separates A and carries a V-connection ρ yielding A-linear maps $\rho(\xi)$ for all $\xi \in L$: owing to the second line in (9)

(8) Note that this argument uses the commutativity of A in an essential way.

we have, for all $\xi \in L$ and $a \in A$, that $(\xi a)X = 0$ for all $X \in V$, hence $\xi a = 0$ since V separates A.

We now present an important explicit class of injective Lie-Cartan pairs (9).

[5] LEMMA. *Let M be a smooth compact n-manifold, with $\mathcal{X}(M)$ the set of smooth vector fields of M* (10), *and $\Lambda^p(M)$ the set of real (complex) smooth p-forms on M: $(\mathcal{X}(M), \Lambda^0 M)$ is a Lie-Cartan pair.*

Examples of left $\Lambda^0(M)$-modules:

(i) $\Lambda^p(M)$, $0 \leqslant p \leqslant n$. *The Lie derivative L is then a flat $\Lambda^p(M)$-connection.*

(ii) *the set $\Gamma(E)$ of smooth sections of any smooth vector bundle E over M. The local $\Gamma(E)$-connections are then exactly the smooth linear connections of E.*

Proof. $(\mathcal{X}(M), \Lambda^0(M))$ is a Lie-Cartan pair as a subpair of $(\mathrm{Der}\,\Lambda^0(M), \Lambda^0(M))$. The other statements are reinterpretations of known definitions.

Our assumption, in the definition of Lie-Cartan pairs (L, A), that A is unital and L unital as a left A-module causes no loss in generality, since we have an «augmentation functor» between the categories of non unital and unital Lie-Cartan pairs:

[6] LEMMA. *Define a* non unital Lie-Cartan pair *as a pair (L, A) satisfying all the requirements in Definition* [1], *except the unital requirements for the algebra A and the left A-module L. The following construction of (L, A) will yield a Lie-Cartan pair* (11), *called* the augmentation of (L, A):

(i) $\tilde{A} = \mathbb{C} \oplus A$ *is the algebra obtained from A by formally adding a unit* $\mathbb{1}$ (12)

(ii) *the products (1) and (2) are respectively defined as*

$$\xi(\alpha \mathbb{1} + a) = \xi a \qquad (26)$$

$$(\alpha \mathbb{1} + a)\xi = \alpha\xi + a\xi \qquad (27)$$

$$, \xi \in L, \alpha \mathbb{1} + a \in \tilde{A}$$

(9) To which the Lie-Cartan pairs owe their name. In fact the compactness assumption in (5) can be removed, using the procedure (6).

(10) In other terms, smooth derivations of $C^\infty(M) = \Lambda^0(M)$.

(11) In the sense of Definition (1).

(12) We recall that $(\alpha, a)(\beta, b) = (\alpha\beta, \alpha b + \beta a + ab)$, $(\alpha, a), (\beta, b) \in C \oplus A$. We then have $\mathbb{1} = (1, 0)$ and $(\alpha, a) = \alpha \mathbb{1} + a, (\alpha, a) \in C \oplus A$.

Moreover, let V be a real (complex) representation vector space of A. The convention

$$(28)\qquad (\alpha\mathbb{1} + a)X = \alpha X + aX,\ \alpha\mathbb{1} + a \in \tilde{A}, X \in V$$

makes V a left A-module, say $\tilde{V}$. Each V-connection then becomes a $\tilde{V}$-connection $\tilde{\rho}$ for the Lie-Cartan pair $(L, \tilde{A})$. If *ρ is local or flat, so is $\tilde{\rho}$.*

Proof. The r.h.s. of both lines in (26) are obviously linear, both in ξ and $\alpha\mathbb{1} + a$. We check the remaining properties:

property (3):

$$(29)\qquad \begin{aligned} &\xi[\alpha\mathbb{1} + a)(\beta\mathbb{1} + b)] \\ &\quad = \xi(\alpha\beta\mathbb{1} + \alpha b + \beta a + ab) = \xi(\alpha b + \beta a + ab) \\ &\quad = \alpha\xi b + \beta\xi a + (\xi a)b + a(\xi b) = (\xi a)(\beta\mathbb{1} + b) + (\alpha\mathbb{1} + \alpha)\xi b \\ &\quad = \{\xi(\alpha\mathbb{1} + \alpha)\}(\beta\mathbb{1} + b) + (\alpha\mathbb{1} + a)\{\xi(\beta\mathbb{1} + b)\} \end{aligned}$$

property (4):

$$(30)\qquad \begin{aligned} [\xi, \eta](\alpha\mathbb{1} + a) &= [\xi, \eta]a = \xi(\eta a) - \eta(\xi\alpha) \\ &= \xi\{\eta(\alpha\mathbb{1} + a)\} - \eta\{\xi(\alpha\mathbb{1} + a)\} \end{aligned}$$

property (1, 5):

$$(31)\qquad \begin{aligned} (\alpha\mathbb{1} + a)\{(\beta\mathbb{1} + b)\xi\} &= (\alpha\mathbb{1} + a)(\beta\xi + b\xi) \\ &= \alpha\beta\xi + \alpha b\xi + \beta a\xi + ab\xi \\ &= (\alpha\beta + \alpha b + \beta a + ab)\xi \\ &= [(\alpha\mathbb{1} + a)(\beta\mathbb{1} + b)]\xi \end{aligned}$$

property (6):

$$(32)\qquad \mathbb{1}\,\xi = \mathbb{1}\cdot\mathbb{1}\,\xi = \mathbb{1}\xi = \xi$$

property (7):

$$(33)\qquad \begin{aligned} \{(\alpha\mathbb{1} + a)\xi\}(\beta\mathbb{1} + b) &= (\alpha\xi + a\xi)(\beta\mathbb{1} + b) = \alpha\xi b + (a\xi)b \\ &= \alpha\xi b + a(\xi b) = (\alpha\mathbb{1} + a)(\xi b) \\ &= (\alpha\mathbb{1} + a)[\xi(\beta\mathbb{1} + b)] \end{aligned}$$

property (8):

$$
\begin{aligned}
[\xi, (\alpha \mathbb{1} + a)\eta] &= [\xi, \alpha\eta + a\eta] = \alpha[\xi, \eta] + [\xi, a\eta] \\
&= \alpha[\xi, \eta] + a[\xi, \eta] + (\xi a)\eta \\
&= (\mathbb{1}\, \alpha + a)[\xi, \eta] + \{\xi(\alpha \mathbb{1} + a)\}\eta.
\end{aligned} \tag{34}
$$

We proved that $(L, \widetilde{A})$ with the products (26) is a Lie-Cartan pair in the sense of Definition [1].

Let now V be a real (complex representation vector space of A, the expressions r.h.s. of (27) is obviously linear in both X and $\alpha\mathbb{1} + a$. Furthermore, we have

$$
\begin{aligned}
(\alpha \mathbb{1} + a)[(\beta \mathbb{1} + b)X] &= (\alpha \mathbb{1} + a)[\beta X + bX) \\
&= \alpha(\beta X + bX) + a(\beta X + bX) \\
&= (\alpha\beta \mathbb{1} + \alpha b + \beta\alpha + ab)X \\
&= [(\alpha \mathbb{1} + a)(\beta \mathbb{1} + b)]X
\end{aligned} \tag{35}
$$

V thus becomes a left $\widetilde{A}$-module $\widetilde{V}$. Let now ρ be a V-connection. We have

$$
\begin{aligned}
\rho(\xi)[\ \alpha \mathbb{1} + a)X] &= \rho(\xi)(\alpha X + aX) = \alpha\rho(\xi)X + a\rho(X) + (\xi a)X \\
&= (\alpha \mathbb{1} + a)\rho(\xi)X + \ \xi(\alpha \mathbb{1} + a)\ X
\end{aligned} \tag{36}
$$

hence ρ is a $\widetilde{V}$-connection, say $\widetilde{\rho}$. If ρ is local we have

$$
\begin{aligned}
\widetilde{\rho}((\alpha \mathbb{1} + a)\xi) &= \rho(\alpha\xi + a\xi) + \alpha\rho(\xi) + a\rho(\xi) \\
&= (\alpha \mathbb{1} + a)\widetilde{\rho}(\xi)
\end{aligned} \tag{37}
$$

thus $\widetilde{\rho}$ is local. Moreover, if ρ is flat, ρ is obviously flat.

We now come to the point of this paper, namely the description, given a Lie-Cartan pair (L, A) and a V-connection ρ (V a linear representation space of A) of an analytical apparatus on the Grassmann space (13) $\Lambda^*(L, V)$ of L^* with values in V, resp. the V-valued Grassmann space $\Lambda^*(L, V)$ over A. We advise the reader to keep in mind that, along with the Lie-Cartan pair (L, A), there comes the depletion $(L : \mathbb{R})$ $((L, \mathbb{C}))$ of (L, A), a degenerate Lie-Cartan pair cf. [4]. Whilst the operators δ_ρ and $\Theta_\rho(\xi)$, $\xi \in L$, pertain to (L, A), the operators δ_0 and $\Theta_0(\xi)$, $\xi \in L$, are in fact their analogues pertaining to $(L, \mathbb{R})$ (to $(L, \mathbb{C})$), whereby the V-connection ρ is replaced by the 0-connection of the corresponding vector space V.

(13) Not Grassmann algebra unless V itself is an algebra. L^* denotes the dual of L.

[7] DEFINITIONS. Let (L, A) be a Lie-Cartan pair and let V be a linear representation space of A (14) equipped with a V-connection ρ. Denoting by $\Lambda^p(L, V)$, resp. $\Lambda^p_A(L, V)$, the set of V-valued alternate p-linear (16) forms, resp. $p - A$ linear forms on L, we set

$$(38) \qquad \begin{cases} \Lambda^*(L, V) = \bigoplus_{p \in N} \Lambda^p(L, V) \\ \Lambda^*_A(L, V) = \bigoplus_{p \in N} \Lambda^p_A(L, V) \end{cases}$$

where $\Lambda^0(L, V) = \Lambda^0_A(L, V) = V$. For $\lambda \in \Lambda^p(L, V)$, and $\xi \in L$, we define

$$(39) \qquad \begin{cases} \{i(\xi)\lambda\}(\xi_1, \ldots, \xi_{p-1}) = \lambda(\xi, \xi_1, \ldots, \xi_{p-1}), \\ \qquad\qquad\qquad\qquad , \xi_i, \xi \in L, i = i \ldots p-1 \\ i(\xi)\lambda = 0 \quad \text{for} \quad \lambda \in \Lambda^0(L, V) \end{cases}$$

further

$$(40) \qquad \delta_\rho \lambda = \delta_0 \lambda + \rho \wedge \lambda$$

and

$$(41) \qquad \Theta_\rho(\xi)\lambda = \Theta_0(\xi)\lambda + \rho(\xi)\lambda$$

where (15)

$$(42) \qquad \begin{cases} (\delta_0 \lambda)(\xi_0, \ldots, \xi_p) = \sum\limits_{0 \leqslant i < j \leqslant p} (-1)^{i+j} \lambda([\xi_i, \xi_j], \xi_0, \ldots, \hat{\xi}_i, \ldots, \hat{\xi}_j, \ldots, \xi_p) \\ (\rho \wedge \lambda)(\xi_0, \ldots, \xi_p) = \sum\limits_{i=0}^{p} (-1)^i \rho(\xi_i) \lambda(\xi_0, \ldots, \hat{\xi}_i, \ldots, \xi_p) \\ \qquad , \xi_0, \xi_1, \ldots, \xi_p \in L, p \geqslant 1 \\ \delta_0 \lambda = \rho \wedge \lambda = 0 \quad \text{for} \quad \lambda \in \Lambda^0(L, V) \end{cases}$$

and

(14) In other terms, a unital left A-module.

(15) The caret ^ indicates omission.

(16) I.e. $\mathbb{R}$-linear ($\mathbb{C}$-linear).

$$(43)\qquad \begin{cases} \{\Theta_0(\xi)\lambda\}(\xi_1, \ldots, \xi_p) = \\ \qquad = -\sum_{i=1}^{p} \lambda(\xi_1, \ldots, \xi_{i-1}, [\xi, \xi_i], \xi_{i+1}, \ldots, \xi_p) \\ \{\rho(\xi)\lambda\}(\xi_1, \ldots, \xi_p) = \rho(\xi)\lambda(\xi_1, \ldots, \xi_p) \\ \qquad , \xi_1, \ldots, \xi_p \in L, p \geqslant 1 \\ \Theta_0(\xi)\lambda = 0 \quad \text{for} \quad \lambda \in \Lambda^0\epsilon(L, V). \end{cases}$$

Finally, for $\lambda \in \Lambda^p(L, V)$, and $a \in A$, we define

$$(44)\qquad \begin{cases} (a\lambda)(\xi_1, \ldots, \xi_p) = a\{\lambda(\xi_1, \ldots, \xi_p)\}, \xi_1, \ldots, \xi_p \in L \\ (a \wedge \lambda)(\xi_0, \ldots, \xi_p) = \sum_{j=0}^{p} (-1)^i (\xi_i a)\lambda(\xi_0, \ldots, \hat{\xi}_i, \ldots, \xi_p), \\ \qquad , \xi_0, \ldots, \xi_p \in L \end{cases}$$

[1.8] THEOREM. *With (L, A), V and ρ as in Definitions* [7] · *and adopting the notation there, we have that*

(i) $\delta_0, \rho\wedge, \delta_\rho$; *$i(\xi)$, $\Theta_0(\xi)$, $\Theta_\rho(\xi)$, $\rho(\xi)$, $\xi \in L$; and $a, a\wedge$, $a \in A$, are linear operators* (16) *of $\Lambda^*(L, V)$, in fact*

$$(45)\qquad \begin{cases} i(\xi)\lambda \epsilon \Lambda^{p-1}(L, V) \\ \Theta_0(\xi)\lambda, \Theta_\rho(\xi)\lambda, \rho(\xi)\lambda, a\lambda \epsilon \Lambda^p(L, V) \\ \delta_0\lambda, \delta_\rho\lambda, \rho \wedge \lambda, a \wedge \lambda \epsilon \Lambda^{p+1}(L, V) \end{cases} \quad \begin{cases} \lambda \epsilon \Lambda^p(L, V), p \in N \\ \xi \in L \\ a \in A \end{cases}$$

Further δ_0, $i(\xi)$, $\Theta_0(\xi)$, $\xi \in L$; and $a, a\wedge$, $a \in A$, are A-linear (17)*; and one has, for $\lambda \in \Lambda^*(L, V)$, $\xi \in L$, $a \in A$*

$$(46)\qquad \rho(\xi)(a\lambda) = a\rho(\xi)\lambda + (\xi a)\lambda$$

$$(47)\qquad \rho \wedge (a\lambda) = a(\rho \wedge \lambda) + a \wedge \lambda$$

$$(48)\qquad \Theta_\rho(\xi)(a\lambda) = a\Theta_\rho(\xi) + (\xi a)\lambda$$

$$(49)\qquad \delta_\rho(a\lambda) = a\delta_\rho\lambda + a \wedge \lambda$$

---

(17) Hence $\Lambda_p(L, V)$ becomes a left A-module with left multiplication (44) and $\Lambda^p_A(L, V)$ a sub left A-module of the latter.

(ii) *one has, for* $\xi, \eta \in L$

(50) $$i(\xi)^2 = 0, \quad i(\xi)\,i(\eta) + i(\eta)\,i(\xi) = 0$$

(51) $$\Theta_\rho(\xi)\,\Theta_\rho(\eta) - \Theta_\rho(\eta)\,\Theta_\rho(\xi) = \Theta_\rho([\xi,\eta]) + \Omega(\xi,\eta)$$

(52) $$\delta_\rho^2 = \Omega\,\wedge$$

(53) $$\delta_\rho i(\xi) + i(\xi)\delta_\rho = \Theta_\rho(\xi)$$

(54) $$i(\xi)\Theta_\rho(\eta) - \Theta_\rho(\eta)\,i(\xi) = i([\xi,\eta])$$

(55) $$\Theta_\rho(\xi)\delta_\rho - \delta_\rho\Theta_\rho(\xi) = \{i(\xi)\Omega\}\wedge = i(\xi)\circ(\Omega\wedge) - (\Omega\,\wedge)\circ i(\xi)$$

(51a) $$\Theta_0(\xi)\,\Theta_0(\eta) - \Theta_0(\eta)\,\Theta_0(\xi) = \Theta_0([\xi,\eta])$$

(52a) $$\delta_0^2 = 0$$

(53a) $$\delta_0 i(\xi) + i(\xi)\delta_0 = \Theta_0(\xi)$$

(54a) $$i(\xi)\Theta_0(\eta) - \Theta_0(\eta)\,i(\xi) = i([\xi,\eta])$$

(55a) $$\delta_0\Theta_0(\xi) = \Theta_0(\xi)\delta_0 = \delta_0\Theta_0(\xi)\delta_0$$

where the operators $\Omega\wedge$ *and* $i(\xi)\Omega\wedge$ *are defined as follows: one has for* $\lambda \in \in \wedge^p(L, V)$, $p \in N$, *and* $\xi_0, \ldots, \xi_{p+2} \in L$ (18)

(56) $$\begin{cases} (\Omega\wedge\lambda)(\xi_1,\ldots,\xi_{p+2}) = \\ \quad = \dfrac{1}{(p+1)!} \displaystyle\sum_{\sigma\in\Sigma_{p+2}} \chi(\sigma)\Omega(\xi_{\sigma 1},\xi_{\sigma 2})\lambda(\xi_{\sigma 3},\ldots,\xi_{\sigma(p+2)}) \\ \{\Omega(\xi,\eta)\lambda\}(\xi_1,\ldots,\xi_p) = \Omega(\xi,\eta)\lambda(\xi_1,\ldots,\xi_p) \end{cases}$$

(57) $$(\{i(\xi)\Omega\}\wedge\lambda)(\xi_0,\ldots,\xi_p) = \sum_{i=0}^{p} (-1)^i\,\Omega_\rho(\xi,\xi_i)\lambda(\xi_0,\ldots,\hat{\xi}_i,\ldots,\xi_p)$$

(iii) *if the connection* ρ *is flat, one has, for all* $\xi \in L$,

(51b) $$\Theta_\rho(\xi)\,\Theta_\rho(\eta) - \Theta_\rho(\eta)\,\Theta_\rho(\xi) = \Theta_\rho([\xi,\eta])$$

(52b) $$\delta_\rho^{\,2} = 0$$

(56b) $$\delta_\rho\Theta_\rho(\xi) = \Theta_\rho(\xi)\delta_\rho = \delta_\rho i(\xi)\delta_\rho$$

(18) Σ_n denotes the set of permutations of the n first integers, with $\chi(\sigma)$ the signature of the permutation σ.

(iv) *if the connection* ρ *is local,* δ_ρ *and* $\Theta_\rho(\xi)$ *(and, of course* $i(\xi)$*), leave* $\Lambda_A^*(L, V)$ *globally invariant:*

$$\rho \text{ local } \Rightarrow \begin{cases} \delta_\rho \lambda \in \Lambda_A^{p+1}(L, V) \\ \\ \Theta_\rho(\xi) \in \Lambda_A^p(L, V) \end{cases}, \begin{cases} \lambda \in \Lambda_A^p(L, V), p \in N \\ \\ \xi \in L. \end{cases} \tag{58}$$

We now assume that V *is equipped with a bilinear* (19) *product*

$$V \times V \ni (u, v) \to u \cdot v \in V. \tag{59}$$

The wedge product $\lambda \wedge \mu$ *of* λ and μ, $\lambda \in \Lambda^p(L, V)$, $\mu \in \Lambda^q(L, V)$, $p, q \in N$ *is then defined as*

$$\begin{cases} (\lambda \wedge \mu)(\xi_1, \ldots, \xi_{p+q}) = \dfrac{1}{p!}\, \dfrac{1}{q!} \displaystyle\sum_{\sigma \in \Sigma_{p+q}} \chi(\sigma) \lambda(\xi_{\sigma 1}, \ldots, \xi_{\sigma p}) \\ \qquad\qquad \cdot\, \mu(\xi_{\sigma(p+1)}, \ldots, \xi_{\sigma(p+q)}) \\ \qquad\qquad , \xi_1, \ldots, \xi_{p+q} \in L \\ (\lambda\mu = \mu\lambda = \mu \cdot \lambda \qquad if \qquad \lambda \in \Lambda^0(L, V)) \end{cases} \tag{60}$$

in particular the operator $\Phi \wedge$, $\Phi \in \Lambda^1(L, V)$ *is given by*

$$\begin{aligned} &(\Phi \wedge \mu)(\xi_1, \ldots, \xi_{p+q}) \\ &= \sum_1^p (-1)^i \Phi(\xi_i) \cdot \lambda(\xi_0, \ldots, \hat{\xi}_i, \ldots, \xi_\mu) \\ &, \lambda \in \Lambda^p(L, V), p \geqslant 1, \xi_0 \cdot \ldots \xi_p \in L \end{aligned} \tag{61}$$

We then have that

(v) *the wedge product has a unique bilinear extension to* $\Lambda^*(L, V)$ *which is* A*-bilinear whenever this is the case for the product*$\cdot$*(in which case* $\Lambda_A^*(L, V)$ *is a graded subalgebra of the -in general not associative-graded algebra* $\Lambda^*(L, V)$*)*

(vi) *we have the derivation and antiderivation relations*

$$i(\xi)(\lambda \wedge \mu) = \{i(\xi)\lambda\} \wedge \mu + (-1)^p \lambda \wedge i(\xi)\mu \tag{62}$$

$$\Theta_0(\xi)(\lambda \wedge \mu) = \{\Theta_0(\xi)\lambda\} \wedge \mu + \lambda \wedge \Theta_0(\xi)\mu, \tag{63}$$

$$\delta_0(\lambda \wedge \mu) = (\delta_0 \lambda) \wedge \mu + (-1)^p \lambda \wedge \delta_0 \mu \tag{64}$$

$$\begin{cases} \lambda \in \Lambda^p(L, V), p \in N \\ \mu \in \Lambda^*(L, V) \\ \xi \in L \end{cases}$$

(19) I.e. $\mathbb{R}$-bilinear (C-bilinear).

which can also be interpreted as commutation or anticommutation relations, e.g. (62) reads

(62a) $$i(\xi)(\lambda\wedge) - (-1)^p(\lambda\wedge)i(\xi) = \{i(\xi)\lambda\}\wedge, \quad \begin{cases} \lambda \in \wedge^p(L, V), p \in N \\ \\ \xi \in L. \end{cases}$$

If in addition $\rho(\xi)$ acts on V as a derivation:

(65) $$\rho(\xi)\cdot\{X\cdot Y\} = \{\rho(\xi)X\}\cdot Y + X\cdot\rho(\xi)Y, \qquad X, Y \in V, \xi \in L$$

$\rho(\xi)$ acting on $\wedge^*(L, V)$ is also a derivation

(66) $$\rho(\xi)(\lambda\wedge\mu) = \rho(\xi)\lambda\wedge\mu + \lambda\wedge\rho(\xi)\mu, \qquad \lambda, \mu \in \wedge^*(L, V), \xi \in L$$

and we have

(67) $$\Theta_\rho(\xi)(\lambda\wedge\mu) = \{\Theta_\rho\xi)\lambda\}\wedge\mu + \lambda\wedge\Theta_\rho\xi)\mu$$

(68) $$\delta_\rho(\lambda\wedge\mu) = (\delta_\rho\lambda)\wedge\mu + (-1)^p\lambda\wedge\delta_\rho\mu$$

$$, \quad \begin{cases} \lambda \in \wedge^p(L, V), p \in N \\ \mu \in \wedge^*(L, V) \\ \xi \in L \end{cases}$$

(vii) *if the product $\cdot$ is associative, so is the wedge product under which $\wedge^*(L, V)$ then becomes a graded associative real (complex) algebra* (20). *One has then* (21)

(69) $$(\lambda\wedge\mu\wedge\nu)(\xi_1, \ldots, \xi_{p+q+r}) = \frac{1}{p!}\,\frac{1}{q!}\,\frac{1}{r!}\sum_{\sigma\in\Sigma_{p+q+r}} \chi(\sigma)$$
$$(\xi_{\sigma 1}, \ldots, \xi_{\sigma p})\cdot\mu(\xi_{\sigma(p+1)}, \ldots, \xi_{\sigma(\mu+q)})\cdot\nu(\xi_{\sigma(p+q+1)})\ldots\xi_{\sigma(p+q+r)})$$
$$, \xi_1, \ldots, \xi_{p+q+r} \in L.$$

(viii) *if V $(= \oplus V^\alpha)$ is a (graded) Lie algebra under the product $\cdot$ (then noted $[\ ,\ ]$, the wedge product (then noted $\wedge$ makes $\wedge^*(L, V)$ a (doubly) graded Lie algebra* (21).

(70) $$\begin{cases} [\mu\wedge\lambda] = (-1)^{pq+1}[\lambda\wedge\mu] \\ (-1)^{pr}[\lambda\wedge[\mu\wedge\nu]] + (-1)^{qp}[\mu\wedge[\nu\wedge\lambda]] + (-1)^{rq}[\nu\wedge[\lambda\wedge\mu]] = 0 \\ \qquad , \lambda \in \wedge^p(L, V), \mu \in \wedge^q(L, V), \nu \in \wedge^r(L, V) \end{cases}$$

---

(20) Anticommutative if the product is commutative.

(21) This holds for a general vector space L — for this results, the fact that L is a Lie algebra is irrelevant.

$$(71)\quad \begin{cases} [\mu \wedge \lambda] = (-1)^{pq+\alpha\beta+1}[\lambda \wedge \mu] \\ (-1)^{pr+\alpha\gamma}[\lambda \wedge [\mu \wedge \nu]] + (-1)^{pq+\beta\alpha}[\mu \wedge [\nu \wedge \lambda]] \\ \qquad\qquad + (-1)^{rq+\gamma\beta}[\nu \wedge [\lambda \wedge \mu]] = 0 \\ \qquad\qquad \lambda \in \Lambda^p(L, V^\alpha), \mu \in \Lambda^q(L, V^\beta), \nu \in \Lambda^r(L, V). \end{cases}$$

Proof. (i) We prove the antisymmetry properties (46). We first check that $\delta_0\lambda$ is alternate, that is, changes sign upon the exchange $\xi_k \langle - \rangle \xi_\ell$, $0 \leqslant k \leqslant \ell < p$. Let A_{ij} be the $i-j$-term in the sum r.h.s. of the first equation (42). For i, j both distinct k, ℓ, A_{ij} changes sign upon exchange $\xi_k \langle - \rangle \xi_\ell$, since λ is alternate. For $\{i, j\} = \{k, \ell\}$, hence $i = j$, $k = \ell$, A_{ij} changes sign upon exchange $\xi_k \langle - \rangle \xi_\ell$ because $[\xi_\ell, \xi_k] = -[\xi_k \cdot \xi_\ell]$. Remain the terms A_{kj}, $j = \ell$, and $A_{i\ell}$ $i = k$. These terms come in pairs $A_{ki}, A_{i\ell}$, $i = k$, $i = \ell$, yielding a contribution

$$(72)\quad \begin{aligned} &(-1)^i\{(-1)^k\lambda([\xi_k, \xi_i], \ldots, \hat{\xi}_k, \ldots, \hat{\xi}_i \ldots) \\ &\qquad + (-1)^i\lambda([\xi_i, \xi_\ell], \ldots, \hat{\xi}_\ell, \ldots, \hat{\xi}_i \ldots) \end{aligned}$$

which changes sign upon the exchange $\xi_k \longleftrightarrow \xi_\ell$.

We now check that $\rho \wedge \lambda$ is alternate. Let A_i be the i-term in the sum r.h.s. of the second equation (42). For $i \neq \ell$, $i \neq k$, A_i changes sign upon exchange $\xi_k \longleftrightarrow \xi_\ell$, since λ is alternate. And the pair A_k, A_ℓ, $k < \ell$, yields a contribution

$$(73)\quad \begin{aligned} &(-1)^k[\rho(\xi_k)\lambda(\xi_0, \ldots, \hat{\xi}_k, \ldots, \xi_\ell, \ldots, \xi_p)] \\ &\qquad -(-1)^{k-1}\rho(\xi_\ell)\lambda(\xi_0, \ldots, \xi_k, \ldots, \hat{\xi}_\ell, \ldots, \xi_p)] \end{aligned}$$

which is turned by the exchange $\xi_k \longleftrightarrow \xi_\ell$ into

$$(74)\quad \begin{aligned} &(-1)^k[\rho(\xi_\ell)\lambda(\xi_0, \ldots, \hat{\xi}_k, \ldots, \xi_k, \ldots, \xi_p) \\ &\qquad -(-1)^{k-\ell}\rho(\xi_k)\lambda(\xi_0, \ldots, \xi_\ell, \ldots, \hat{\xi}_\ell, \ldots, \xi_p)] \\ &= (-1)^k[(-1)^{k-\ell-1}\rho(\xi_\ell)\lambda(\xi_0, \ldots, \xi_k, \ldots, \hat{\xi}_\ell, \ldots, \xi_p) \\ &\qquad -\rho(\xi_k)\lambda(\xi_0, \ldots, \hat{\xi}_k, \ldots, \xi_\ell, \ldots, \xi_p)], \end{aligned}$$

thus changes sign.

$\delta_\rho\lambda$ is then alternate as the sum $\delta_0\lambda + \rho \wedge \lambda$. And $i(\xi)\lambda$ and $\rho(\xi)\lambda$, $\xi \in L$, are evidently alternate.

We check that $\Theta_0(\xi)$, $\xi \in L$, is alternate, looking at the exchange $\xi_k \longleftrightarrow \xi_\ell$. Let B_i be the i-term r.h.s. of the first eqution (53). For $i \neq k$, $i \neq \ell$, B_i changes sign upon the exchange $\xi_k \longleftrightarrow \xi_\ell$. And this exchange turns B_k into $-B_\ell$ and B_ℓ into $-B_k$.

We now prove properties (46) through (50).

Since δ_0, $i(\xi)$, and $\Theta_0(\xi)$ act only on the arguments of λ, they are obviously A-linear, and so are a and $a\wedge$ since A in commutative: we prove (46) and (47) (from which (48) and (49) immediately follow): from (29) we have:

$$\begin{aligned}\rho(\xi)(a\lambda)(\xi_1,\ldots,\xi_p) &= \rho(\xi)\{a\lambda(\xi_1,\ldots,\xi_p)\}\\ &= a\rho(\xi)\lambda(\xi_1,\ldots,\xi_p) + (\xi a)\lambda(\xi_1,\ldots,\xi_p)\\ &= \{(a\rho(\xi)\lambda + (\xi a)\lambda\}(\xi_1,\ldots,\xi_p)\end{aligned} \tag{75}$$

$$\begin{aligned}(\rho\wedge a\lambda)(\xi_0,\ldots,\xi_p) &= \sum_{i=0}^{p}(-1)^i\rho(\xi_i)a\lambda(\xi_0,\ldots,\hat{\xi}_i,\ldots,\xi_p)\\ &= a\sum_{i=0}^{p}(-1)^i\rho(\xi_i)\lambda(\xi_0,\ldots,\hat{\xi}_i,\ldots,\xi_p)\\ &\quad+\sum_{i=0}^{p}(-1)^i(\xi_i a)\lambda(\xi_0,\ldots,\hat{\xi}_i,\ldots,\xi_p).\end{aligned} \tag{76}$$

(ii) Relations (50) and (51a) (hence (51)) are straightforward, iterating definitions (39), (43) and using for the latter, the Jacobi identity. We prove (53): we have, for $\lambda\in\Lambda^p(L,V)$, $\xi,\xi_1,\ldots,\xi_p\in L$

$$\begin{aligned}\{\delta_\rho i(\xi)\lambda\}(\xi_1,\ldots;\xi_p) &= \sum_{i=1}^{p}(-1)^{i+1}\rho(\xi_i)\lambda(\xi,\xi_1,\ldots,\hat{\xi}_i,\ldots,\xi_p)\\ &\quad+\sum_{1\leqslant i<j\leqslant p}(-1)^{i+j}\lambda(\xi,[\xi_i,\xi_j],\xi_1,\ldots,\hat{\xi}_i,\ldots,\hat{\xi}_j,\ldots,\xi_p)\end{aligned} \tag{77}$$

$$\begin{aligned}\{i(\xi)\delta_\rho\lambda\}(\xi_1,\ldots,\xi_p) &= \rho(\xi)\lambda(\xi_1,\ldots,\xi_p)\\ &\quad+\sum_{i=1}^{p}(-1)^i\rho(\xi_i)\lambda(\xi,\xi_1,\ldots,\hat{\xi}_i,\ldots,\xi_p)\\ &\quad+\sum_{i=1}^{p}(-1)^i\lambda([\xi,\xi_i],\xi_1,\ldots,\hat{\xi}_i,\ldots,\xi_p)\\ &\quad+\sum_{1\leqslant i<j\leqslant p}(-1)^{i+j}\lambda([\xi_i,\xi_j],\xi,\ldots,\hat{\xi}_i,\ldots,\hat{\xi}_j,\ldots,\xi_p).\end{aligned} \tag{78}$$

Upond addition the first term r.h.s. of (77), resp. the second, concels with the second term r.h.s. of (78), resp. with the last, the remaining terms yielding $\Theta_\rho(\xi)\lambda$: we proved (53). A straightforward application of definitions (39), (43) yields (54a) and (54) ($i(\xi)$ and $\rho(\eta)$ commute). We now prove that δ_0 and $\Theta_0(\xi)$ commute for each $\xi \in L$. We have, for $\lambda \in \Lambda^p(L, V)$ and $\xi_0, \ldots, \xi_\rho \in L, p > 1$

(79)
$$-\{\Theta_0(\xi)\lambda\}(\xi_1, \ldots, \xi_p) = \lambda([\xi, \xi_1], \xi_2, \ldots, \xi_p) + \sum_{k=2}^{p} \lambda(\xi_1, \ldots, [\xi, \xi_k], \ldots, \xi_p)$$

hence

$$-\{\delta_0\Theta_0(\xi)\lambda\}(\xi_0, \ldots, \xi_p) =$$

(80)
$$\sum_{0<i<j<p} (-1)^{i+j}\{\lambda([\xi, [\xi_i, \xi_j]], \xi_0, \ldots, \hat{\xi}_i, \ldots, \hat{\xi}_j, \ldots, \xi_p) + \sum_{\substack{k=0\\ k\neq i\\ k\neq j}}^{p} \lambda([\xi_i, \xi_j], \xi_0, \ldots, \hat{\xi}_i, \ldots, \hat{\xi}_j, \ldots, [\xi, \xi_k], \ldots, \xi_p)\}$$

Using there $[\xi, [\xi_i, \xi_j]] = [[\xi, \xi_i], \xi_j] + [\xi_i, [\xi, \xi_j]$ (Jacobi identity) we see that the r.h.s. yields $-\{\Theta_0(\xi)\delta_0\lambda\}(\xi_1, \ldots, \xi_p)$: we proved commutativity of δ_0 and Θ_0 on $\Lambda^p(L, V)$, $p > 1$. This extends to $p = 0$, since δ_0 and $\Theta_0(\xi)$ both vanish on $\Lambda^0(L, V)$.

We now prove (55).

With brackets indicating commutators, we have, for $\xi \in L$.

(81)
$$[\Theta_\rho(\xi), \delta_\rho] = [\Theta_0(\xi) + \rho(\xi), \rho \wedge]$$

since δ_0 commutes with $\Theta_0(\xi)$ (as we just showed) and with $\rho(\xi)$, $\xi \in L$, as is obvious. Now we have, on the one hand, for $\lambda \in \Lambda^p(L, V)$ and $\xi_0, \ldots, \xi_p \in L$.

(82)
$$\{\rho(\xi)(\rho \wedge \lambda) - \rho \wedge \rho(\xi)\lambda\}(\xi_0, \ldots, \xi_p) = \sum_{i=0}^{n} (-1)^i[\rho(\xi), \rho(\xi_i)]\lambda(\xi_0, \ldots, \xi_i, \ldots, \xi_p)$$

and on the other nand

$$-\{\rho\wedge\Theta_0(\xi)\lambda\}(\xi_0,\dots,\xi_p)$$

$$(83)\quad \begin{aligned} &=\sum_{j=0}^{p}\Bigg\{\sum_{i=0}^{j-i}(-1)^i\rho(\xi_i)\lambda(\xi_0,\dots,\hat{\xi}_i,\dots[\xi,\xi_j],\dots\xi_p)\\ &+(-1)^j\rho(\xi_j)\lambda(\xi_0,\dots,\xi_{j-1}[\xi,\xi_{j+1}]\xi_{j+2},\dots,\xi_p)\\ &+\sum_{i=j+1}^{p}(-1)^i\rho(\xi_i)\lambda(\xi_0,\dots,[\xi,\xi_j],\dots,\hat{\xi}_i,\dots,\xi_p)\Bigg\}\end{aligned}$$

$$\begin{aligned} &=\sum_{0\leqslant i<j\leqslant p}(-1)^i\rho(\xi_i)\lambda(\xi_0,\dots,\hat{\xi}_i,\dots[\xi,\xi_j],\dots,\xi_p)\\ &+\sum_{0\leqslant j<i\leqslant p}\rho(\xi_i)\lambda(\xi_0,\dots,[\xi,\xi_j],\dots\hat{\xi}_i,\dots,\xi_p)\end{aligned}$$

$$-\{\Theta_\rho(\xi)\rho\wedge\lambda\ (\xi_0,\dots\xi_p)$$

$$(84)\quad \begin{aligned} &=\sum_{i=0}^{p}(-1)^i\{\rho(\xi_i)\Bigg\{\sum_{j=0}^{i-1}\lambda(\xi_0,\dots,[\xi,\xi_j],\dots,\hat{\xi}_i,\dots,\xi_p)\\ &+\rho([\xi,\xi_i])\,\lambda(\xi_0,\dots,\hat{\xi}_i,\dots,\xi_p)\\ &+\sum_{j=i+1}^{p}\rho(\xi_i)\lambda(\xi_0,\dots,\hat{\xi}_i,\dots,[\xi,\xi_j],\dots,\xi_p)\Bigg\}\end{aligned}$$

$$=-\{\rho\wedge\Theta_0(\xi)\lambda\}(\xi_0,\dots\xi_p)+\sum_{i=0}^{p}(-1)^i\rho([\xi,\xi_i])\lambda(\xi_0,\dots,\xi_i,\dots,\xi_p)\},$$

taking account of (81)

$$(85)\qquad \{[\Theta_\rho(\xi),\delta_\rho]\lambda\}(\xi_0,\dots,\xi)=\sum_{i=0}^{p}(-1)^i\Omega(\xi,\xi_i)\lambda(\xi_0,\dots,\hat{\xi}_i,\dots,\xi_p)$$

which is the first equation (55), cf. (57). The second equation corresponds to the special case $p=2$ of the general relation (62), which is proved below (in fact $\Omega(\xi,\xi')$ belongs to End V, not to V but this does not change anything to the proof).

We now deduce (52) from (53) and (55) [22]. First multiplying (53) right and left by δ_ρ, and taking the difference leads to the relation

$$[\Theta_\rho(\xi), \delta_\rho] = [i(\xi), \delta_\rho^2], \quad \xi \in L. \tag{86}$$

Equating the l.h.s. of this with that of (55) now yields

$$i(\xi)(\delta_\rho^2 \Omega \wedge) - (\delta_\rho^2 \Omega \wedge) i(\xi) = 0, \quad \xi \in L. \tag{87}$$

We now prove (52) by showing that

$$(\delta_\rho^2 - \Omega \wedge)\lambda = 0, \qquad \lambda \in \wedge^p(L, V) \tag{88}$$

by induction with respect to p. If we know this for $p-1$, we have, by (87), for $\lambda \in \wedge^p(L, V)$

$$i(\xi)(\delta_\rho^2 \lambda - \Omega \wedge \lambda) = (\delta_\rho^2 - \Omega \wedge) i(\xi) \lambda = 0 \tag{89}$$

for each $\xi \in L$, whence the result for p. Now, for $\lambda_0 \in \wedge^0(L, V)$ we have

$$(\delta_\rho \lambda_0)(\xi) = \rho(\xi)\lambda_0 \tag{90}$$

thus

$$\begin{aligned} &(\delta_\rho^2 \lambda_0)(\xi_0, \xi_1) \\ &= \{\rho(\xi_0)\rho\xi_1) - \rho(\xi_1)\rho(\xi_0) - \rho([\xi_0, \xi_1])\}\lambda_0 \\ &= \Omega(\xi_0, \xi_1)\lambda_0 = (\Omega \wedge \lambda_0)(\xi_0, \xi_1). \end{aligned} \tag{91}$$

We now prove (52a) through (55a) from (52) through (55). Since the latter relations hold for any Lie-Cartan pair (L, V) and any V-connection ρ, they hold in particular for the depletion (L, R) ((L, C) of (L, A) and the (then licit) V-connection $\rho = 0$, for which $\delta_\rho \to \delta_0$, $\Theta_\rho \to \Theta_0$ and $\Omega \to 0$). Equation ($\cdot$) then yields ($\cdot a$).

(iii) If ρ is flat, $\Omega = 0$, (52) and (53) then become (52b) and (53b).

(iv) We assume ρ local and λ $p - A$-linear and prove that $\delta_\rho \lambda$ is $(p+1) - A$-linear by checking A-linearity with respect to ξ_k, $0 \leqslant k \leqslant 1$: Denote as above by A_{ij} the $i - j$ term in the sum r.h.s. of the first equation (42) and by A_i the i-term r.h.s. of the second equation (42). Now all the A_{ij} with $i \neq k$, $j \neq k$, and A_k, are A-linear by assumption. Remains the sum of terms A_{kj}, A_{ik} and A_i, A_j. We have using (8), (9), for $j > k$:

---

[22] The forth coming, indirect, argument is motivated by the involved nature of a direct calculation of δ_ρ^2.

$$
\begin{aligned}
&(A_{kj} + A_j)(\xi_0, \dots, \xi_{k-1}, a\xi_k, \xi_{k+1}, \dots, \xi_p) \\
&= (-1)^{k+j}\lambda([a\xi_k, \xi_j], \xi_0, \dots, \hat{\xi}_k, \dots, \hat{\xi}_j, \dots, \xi_p) \\
&+ (-1)^j \rho(\xi_j)\lambda(\xi_0, \dots, a\xi_k, \dots, \hat{\xi}_j, \dots, \xi_p) \\
&= (-1)^{k+j} a\lambda([\xi_k, \xi_j], \xi_0, \dots, \hat{\xi}_k, \dots, \hat{\xi}_j, \dots, \xi_p) \\
&- (-1)^{k+j}(\xi_j a)\lambda(\xi_k, \xi_0, \dots, \hat{\xi}_k, \dots, \hat{\xi}_j, \dots, \xi_p) \\
&+ (-1)^j a\rho(\xi_j)\lambda(\xi_0, \dots, \hat{\xi}_j, \dots, \xi_p) \\
&+ (-1)^j (\xi_j a)\lambda(\xi_0, \dots, \hat{\xi}_j, \dots, \xi_p) \\
&= a(A_{kj} + A_j)(\xi_0, \dots, \xi_p)
\end{aligned} \tag{92}
$$

and, for, $i < k$

$$
\begin{aligned}
&(A_{ik} + A_i)(\xi_0, \dots, \xi_{k-1}, a\xi_k, \xi_{k+1}, \dots, \xi_p) \\
&= (-1)^{i+k}\lambda([\xi_i, a\xi_k], \xi_0, \dots, \hat{\xi}_i, \dots, \hat{\xi}_k, \dots, \xi_p) \\
&+ (-1)^i \rho(\xi_i)\lambda(\xi_0, \dots, a\xi_k, \dots, \hat{\xi}_i, \dots, \xi_p) \\
&= (-1)^{i+k} a\lambda([\xi_i, \xi_k], \xi_0, \dots, \hat{\xi}_i, \dots, \hat{\xi}_k, \dots, \xi_p) \\
&+ (-1)^{i+k}(\xi_i a)\lambda(\xi_k, \xi_0, \dots, \hat{\xi}_i, \dots, \hat{\xi}_k, \dots, \xi_p) \\
&+ (-1)^i a\rho(\xi_i)\lambda(\xi_0, \dots, \hat{\xi}_i, \dots, \xi_p) \\
&+ (-1)^i (\xi_i a)\lambda(\xi_0, \dots, \hat{\xi}_i, \dots, \xi_p) \\
&= a(A_{ik} + A_i)(\xi_0, \dots, \xi_p)
\end{aligned} \tag{93}
$$

We now prove that $i(\xi)$, $\xi \in L$, is a graded derivation. We have, for $\lambda \in \Lambda^p(L, V)$, $\mu \in \Lambda^q(L, V)$, $\xi_1, \dots, \xi_{p+1}$,

$$
\begin{aligned}
&\{(i(\xi)\lambda \wedge \mu + (-1)^p \lambda \wedge i(\xi)\mu\}(\xi_1, \dots, \xi_{p+q-1}) \\
&= \frac{1}{(p-1)!q!} \sum_{\sigma \in \Sigma_{p+q-1}} \chi(\sigma)\lambda(\xi, \xi_{\sigma 1}, \dots, \xi_{\sigma(p-1)}) \cdot \mu(\xi_{\sigma p}, \dots, \xi_{\sigma(p+q-1)}) \\
&+ \frac{(-1)^p}{p!(q-1)!} \sum_{\sigma \in \Sigma_{p+q-1}} \chi(\sigma)\lambda(\xi_{\sigma 1}, \dots, \xi_{\sigma p}) \cdot \mu(\xi, \xi_{\sigma(p+1)} \cdots \xi_{\sigma(p+q-1)}) \\
&= \frac{1}{p!q!} \sum_{\sigma \in \Sigma_{p+q-1}} \chi(\sigma)\Big\{\sum_{i=0}^{p} (-1)^{i-1}\lambda(\xi_{\sigma 1}, \dots, \xi_{\sigma(i-1)}\xi\xi_{\sigma(i+1)}, \dots, \xi_{\sigma(p-1)}) \\
&\qquad \cdot \mu(\xi_{\sigma p}, \dots, \xi_{\sigma(p+q-1)})
\end{aligned} \tag{94}
$$

$$+\sum_{i=0}^{q}(-1)^{i-1}\lambda(\xi_{\sigma 1},\dots,\xi_{\sigma p})\cdot$$

(94)
$$\cdot\,\mu(\xi_{\sigma(p+1)},\dots,\xi_{\sigma(p+i+1)}\xi\xi_{\sigma(p+i+2)},\dots,\xi_{\sigma(p+q-1)})$$
$$=(\lambda\wedge\mu)(\xi,\xi_1,\dots,\xi_{p+q-1})=i(\xi)\{\lambda\wedge\mu\}(\xi_1,\dots,\xi_{p+q-1}).$$

We now prove that $\Theta_0(\xi)$, $\xi\in L$, is a graded derivation.

We first note that, for $\lambda\in\Lambda^p(L,V)$, $\mu\in\Lambda^q(L,V)$

$$\lambda\wedge\mu=\frac{1}{p!\,q!}\sum_{\sigma\in\Sigma_{p+q}}\sigma(\lambda\otimes\mu) \tag{95}$$

where

$$(\sigma\Psi)(\xi_1,\dots,\xi_n)=\Psi(\xi_{\sigma 1},\dots,\xi_{\sigma n}),\quad \Psi\in(L^*)^{\otimes n}\otimes V,\ \xi_1,\dots,\xi_n\in L \tag{96}$$

and

$$(\lambda\otimes\mu)(\xi_1,\dots,\xi_{p+q})=\lambda(\xi_1,\dots,\xi_p)\cdot\mu(\xi_{p+1},\dots,\xi_{p+q}) \tag{97}$$
$$\xi_1,\dots,\xi_{p+q}\in L.$$

Now, defining $\Theta_0(\xi)$ on arbitrary p-linear forms λ by the same formula as in (43), it is obvious that we have

$$\Theta_0(\xi)(\lambda\otimes\mu)=\Theta_0(\xi)\lambda\otimes\mu+\lambda\otimes\Theta_0(\xi)\mu,\quad \lambda,\mu\in\Lambda^*(L,V) \tag{98}$$

which yields (63) upon application of $\frac{1}{p!q!}\sum_{\sigma\in\Sigma_{p+q}}$ on both sides, owing to (95) and the fact, obvious from the definition of $\Theta_0(\xi)$, that $\Theta_0(\xi)$ commutes with the operators σ.

If we assume that $\rho(\xi)$ acting on V, $\xi\in L$, is a derivation for the product: $V\times V\to V$, it is obvious from the second equation (43) and the definition (60) of the wedge product, that $\rho(\xi)$ acting on $\Lambda^*(L,V)$ is a derivation for this wedge product (observe that $\rho(\xi)$ commutes with the operators σ: hence in that case we have the derivation property (67) for $\Theta_\rho(\xi)$).

We now show that (68) follows from (53) and from the derivation properties (62) and (67) of $i(\xi)$ and $\Theta_\rho(\xi)$ [23]. We prove (68) by induction on the order $p+q$ of $\lambda\wedge\mu$.

The property clearly holds for $p=0$, $q=0$, since, then, we have $\delta_\rho\lambda=\delta_\rho\mu=0$

[23] Hence that (64) follows from (53a), (62) and (63) (take $\rho=0$).

$$+(-1)^{rq+\gamma\beta}\sum_{\sigma\in\Sigma_{p+q+r}}\chi(\sigma)[\nu(\xi_{\sigma 1},\dots,\xi_{\sigma p})$$

(104)

$$\wedge[\lambda(\xi_{\sigma(p+1)},\dots,\xi_{\sigma(p+q)}\wedge\mu(\xi_{\sigma(p+q+1)},\dots,\xi_{\sigma(p+q+r)})]]$$

The second and third term r.h.s. are respectively equal to

$$(-1)^{qp+\beta\alpha}\chi(\tau)\sum_{\sigma\in\Sigma_{p+q+r}}\chi(\sigma)[\mu(\xi_{\sigma(p+1)},\dots,\xi_{\sigma(p+q)}$$

(105)

$$\wedge[\nu(\xi_{\sigma(p+q+1)},\dots,\xi_{\sigma(p+q+r)})\wedge\lambda(\xi_{\sigma 1},\dots,\xi_{\sigma p})]]$$

and

$$(-1)^{rq+\nu\beta}\chi(\tau')\sum_{\sigma\in\Sigma_{p+q+r}}\chi(\sigma)[\nu(\xi_{\sigma(p+q+1)},\dots,\xi_{\sigma(p+q+r)})$$

(106)

$$\wedge[\lambda(\xi_{\sigma 1},\dots,\xi_{\sigma p})\wedge\mu(\xi_{\sigma(p+1)},\dots,\xi_{\sigma(p+q)})]]$$

with τ and τ' the permutation

$$\begin{cases}\tau(1,\dots,p+q+r)=(p+1,\dots,p+q+r,1,\dots,p)\\ \tau'(1,\dots,p+q+r)=(p+q+1,\dots,p+q+r,1,\dots,p+q)\end{cases}\tag{107}$$

with signatures $\chi(\tau)=(-1)^{p(q+r)}$, $\chi(\tau')=(-1)^{r(p+q)}$: the l.h.s of (103) then vanishes owing to the assumption that V is a graded Lie Algebra, hence one has

$$\begin{cases}(-1)^{\alpha\gamma}[u\wedge[v\wedge w]]+(-1)^{\beta\alpha}[v\wedge[w\wedge u]]+(-1)^{\alpha\beta}[w\wedge[u\wedge v]=0\\ u\in V^\alpha,\quad v\in V^\beta,\quad w\in V^\gamma.\end{cases}\tag{108}$$

We now apply Theorem [8] to the examples [6]: vector-valued cohomology of Lie algebras (24), exterior derivative on manifolds, exterior covariant derivative on smooth vector bundles:

[9] COROLLARY. *Let (L,A) be a Lie-Cartan pair* (24), *with V a left-A module. And let ρ be a representation of L on V:*

(24) Real or complex i.e. L is a real (complex) Lie algebra, A is a unital commutative real (complex) algebra, L is a unital left A-module, and one has $(a\xi)b=a(\xi b)$ and $\xi(ab)=(\xi a)b+$ $+a(\xi b)$, $\xi\in L, a,b\in A$, $[\xi,a\eta]=a[\xi,\eta]+(\xi a)\eta$.

(viii) We have, for $\lambda \in \Lambda^p(L, V^\alpha)$, $\mu \in \Lambda^q(L, V^\beta)$, p, q, α, $\beta \in N$, and $\xi_1, \dots, \xi_{p+q} \in L$:

$$
\begin{aligned}
&[\mu \wedge \lambda](\xi_1, \dots, \xi_{p+q})\\
&= \frac{1}{p!q!} \sum_{\sigma \in \Sigma_{p+q}} \chi(\sigma)[\mu(\xi_{\sigma 1}, \dots, \xi_{\sigma p}) \wedge \lambda(\xi_{\sigma(p+1)}, \dots, \xi_{\sigma(p+q)})]\\
&= \frac{(-1)^{pq}}{p!q!} \sum_{\sigma \in \Sigma_{p+q}} \chi(\sigma)[\lambda(\xi_{\sigma(p+1)}, \dots, \xi_{\sigma(p+q)} \wedge \mu(\xi_{\sigma 1}, \dots, \xi_{\sigma p})]\\
&= (-1)^{pq+\alpha\beta+1}[\lambda \wedge \mu](\xi_1, \dots, \xi_{p+q})
\end{aligned}
\tag{101}
$$

and where we used the fact that

$$
[v, u] = (-1)^{\alpha\beta+1}[u, v], \quad u \in V^\alpha, \quad v \in V^\beta, \tag{102}
$$

part of our assumption that V is a graded Lie Algebra.

We now prove (71), from which (70) will follow as the special case corresponding to a trivial grading of $V : V = V^0$. We have for $\lambda \in \Lambda^p(L, V^\alpha)$, $\mu \in \Lambda^q(L, V^\beta)$, $\nu \in \Lambda^r(L, V^\gamma)$, p, q, r, α, β, $\gamma \in N$; and $\xi_1 \cdot \dots \cdot \xi_{p+q+r \in L}$, according to a previous calculation

$$
\begin{aligned}
&[[\lambda \wedge \mu](\xi_1, \dots, \xi_{p+q+r})\\
&= \frac{1}{p!q!r!} \sum_{\sigma \in \Sigma_{p+q+r}} \chi(\sigma)[[\lambda(\xi_{\sigma 1}, \dots, \xi_{\sigma p})\\
&\wedge \mu(\xi_{\sigma(p+1)}, \dots, \xi_{\sigma(p+q)}] \wedge \lambda(\xi_{\sigma(p+q+1)}, \dots, \xi_{\sigma(p+q+r)}],
\end{aligned}
\tag{103}
$$

thus, denoting by Ξ the l.h.s. of the second equation (71)

$$
\begin{aligned}
&p!q!r!\,\Xi(\xi_1, \dots, \xi_{p+q+r})\\
&= (-1)^{pq+\alpha\beta} \sum_{\sigma \in \Sigma_{p+q+r}} \chi(\sigma)[\lambda(\xi_{\sigma 1}, \dots, \xi_{\sigma p})\\
&[\mu(\xi_{\sigma(p+1)}, \dots, \xi_{\sigma(p+q)} \wedge \nu(\xi_{\sigma(p+q+1)} \cdot \dots \cdot \xi_{\sigma(p+q+r)}]]\\
&+ (-1^{qp+\beta\alpha} \sum_{\sigma \in \Sigma_{p+q+r}} \chi(\sigma)[\mu(\xi_{\sigma 1}, \dots, \xi_{\sigma p}) \wedge\\
&[\nu(\xi_{\sigma(p+1)}, \dots, \xi_{\sigma(p+q)} \wedge \lambda(\xi_{\sigma(p+q+1)}, \dots, \xi_{\sigma(p+q+r)}]]
\end{aligned}
\tag{104}
$$

$$\begin{aligned}&+(-1)^{rq+\gamma\beta}\sum_{\sigma\in\Sigma_{p+q+r}}\chi(\sigma)[\nu(\xi_{\sigma 1},\ldots,\xi_{\sigma p})\\ &\wedge[\lambda(\xi_{\sigma(p+1)},\ldots,\xi_{\sigma(p+q)}\wedge\mu(\xi_{\sigma(p+q+1)},\ldots,\xi_{\sigma(p+q+r)})]]\end{aligned}\tag{104}$$

The second and third term r.h.s. are respectively equal to

$$\begin{aligned}&(-1)^{qp+\beta\alpha}\chi(\tau)\sum_{\sigma\in\Sigma_{p+q+r}}\chi(\sigma)[\mu(\xi_{\sigma(p+1)},\ldots,\xi_{\sigma(p+q)})\\ &\wedge[\nu(\xi_{\sigma(p+q+1)},\ldots,\xi_{\sigma(p+q+r)})\wedge\lambda(\xi_{\sigma 1},\ldots,\xi_{\sigma p})]]\end{aligned}\tag{105}$$

and

$$\begin{aligned}&(-1)^{rq+\nu\beta}\chi(\tau')\sum_{\sigma\in\Sigma_{p+q+r}}\chi(\sigma)[\nu(\xi_{\sigma(p+q+1)},\ldots,\xi_{\sigma(p+q+r)})\\ &\wedge[\lambda(\xi_{\sigma 1},\ldots,\xi_{\sigma p})\wedge\mu(\xi_{\sigma(p+1)},\ldots,\xi_{\sigma(p+q)})]]\end{aligned}\tag{106}$$

with τ and τ' the permutation

$$\begin{cases}\tau(1,\ldots,p+q+r)=(p+1,\ldots,p+q+r,1,\ldots,p)\\ \tau'(1,\ldots,p+q+r)=(p+q+1,\ldots,p+q+r,1,\ldots,p+q)\end{cases}\tag{107}$$

with signatures $\chi(\tau)=(-1)^{p(q+r)}$, $\chi(\tau')=(-1)^{r(p+q)}$: the l.h.s of (103) then vanishes owing to the assumption that V is a graded Lie Algebra, hence one has

$$\begin{cases}(-1)^{\alpha\gamma}[u\wedge[v\wedge w]]+(-1)^{\beta\alpha}[v\wedge[w\wedge u]]+(-1)^{\alpha\beta}[w\wedge[u\wedge v]=0\\ u\in V^{\alpha},\quad v\in V^{\beta},\quad w\in V^{\gamma}.\end{cases}\tag{108}$$

We now apply Theorem [8] to the examples [6]: vector-valued cohomology of Lie algebras (24), exterior derivative on manifolds, exterior covariant derivative on smooth vector bundles:

[9] COROLLARY. *Let (L,A) be a Lie-Cartan pair* (24), *with V a left-A module. And let ρ be a representation of L on V:*

---

(24) Real or complex i.e. L is a real (complex) Lie algebra, A is a unital commutative real (complex) algebra, L is a unital left A-module, and one has $(a\xi)b=a(\xi b)$ and $\xi(ab)=(\xi a)b+$ $+a(\xi b)$, $\xi\in L, a, b\in A$, $[\xi, a\eta]=a[\xi,\eta]+(\xi a)\eta$.

(109) $\rho([\xi, \eta]) = [\rho(\xi), \rho(\eta)], \qquad \xi, \eta \in L$

which is local

(110) $\rho(a\xi) = a\rho(\xi), \qquad \xi \in L, \quad a \in A.$

If we define, for $\lambda \in \Lambda^p(L, V), p \in N, \xi \in L,$ *and* $\xi_0, \ldots, \xi_p \in L$

$$
(111) \qquad \begin{cases}
(\delta\lambda)(\xi_0,\dots,\xi_p) = \sum_{i=0}^{p} (-1)^i \rho(\xi_i)\lambda(\xi_0,\dots,\hat{\xi}_i,\dots,\xi_p) \\
\quad + \sum_{0 \leq i < i \leq p} (-1)^{i+j} \lambda([\xi_i, \xi_j], \xi_0,\dots,\hat{\xi}_i,\dots,\hat{\xi}_j,\dots,\xi_p) \\
\{\Theta(\xi)\lambda\}(\xi_1,\dots,\xi_p) = \sum_{i=0}^{p} \rho(\xi)\lambda(\xi_1,\dots,\xi_p) \\
\quad - \sum_{i=0}^{p} \lambda(\xi_1,\dots,[\xi, \xi_i], \dots, \xi_p) \\
\{i(\xi)\lambda\}(\xi_1,\dots,\xi_{p-1}) = \lambda(\xi, \xi_1,\dots,\xi_p)
\end{cases}
$$

we obtain linear operators of respective grades $1, 0,$ *and* -1 *on the graded vector space* $\Lambda^*(L, A) = \bigoplus_{p \in \mathbb{N}} \Lambda^p(L, A)$ *with the properties* (25)

$$
(112) \qquad \begin{cases}
\delta(a\lambda) = a\delta\lambda + a \wedge \lambda \\
\Theta(\xi)(a\lambda) = a\Theta(\xi) + (\xi a)\lambda \\
i(\xi)(a\lambda) = a\, i(\xi)\lambda
\end{cases}
$$

(113) $\delta^2 = 0$

(114) $i(\xi)^2 = 0$

(115) $\delta \cdot i(\xi) + i(\xi) \cdot \delta = \Theta(\xi)$

(115bis) $\Theta(\xi)\,\Theta(\eta) - \Theta(\eta)\,\Theta(\xi) = \Theta([\xi, \eta])$

(116) $\delta \cdot \Theta(\xi) = \Theta(\xi) \cdot \delta = \delta \cdot i(\xi) \cdot \delta$

(117) $i(\xi) \cdot \Theta(\eta) - \Theta(\eta)\, i(\xi) = i([\xi, \eta]).$

Consequently, $(\Lambda^*(L, V), \delta)$ *is a cochain complex whose cohomology is denoted*

---

(25) $a \wedge \lambda$ is defined in (55).

$H^*(L, V)$ *and called the* A-local cohomology of L with values in V. *Note that* $\Theta(\xi)$, $\xi \in L$, *turns cocycles into coboundaries:*

$$\Theta(\xi)\, Z_A^p(L, V) \in B_A^p(L, V) \tag{118}$$

In the particular case $A = \mathbb{R}$ ($A = \mathbb{C}$) *trivially acted upon by* L (26), *one gets the usual notion of the* Chevalley cohomology of the Lie algebra L with values in the representation vector space $V : H^*(L, V)$.

If, furthermore, $V = \mathbb{R}$ ($V = \mathbb{C}$), *hence* ρ *vanishes, one gets the usual* cohomology $H^*(L)$ of the Lie-algebra L, *which is an algebra for the wedge product. In that case* $\Lambda^*(L, \mathbb{R})$ ($\Lambda^*(L, \mathbb{C})$ *is the usual* Grassmann algebra over L, *of which* δ, $i(\xi)$, $\xi \in L$, *are antiderivations* (27) *of respective grades* 1 *and* -1, $\Theta(\xi)$, $\xi \in L$, *being a derivation of grade zero.*

[10] COROLLARY. *Let* M *be a smooth* n*-manifold, with* $\mathscr{X}(M)$ *the Lie algebra of smooth real (complex) vector fields on* M, *and* $\Lambda^p(M)$, $0 \leqslant p \leqslant n$, *the set of smooth* p*-forms on* M. *If we define, for* $\lambda \in \Lambda^p(M)$, *and* $\xi_0, \ldots, \xi_p \in \mathscr{X}(M)$

$$\left\{\begin{aligned}
&(d\lambda)(\xi_0 \cdot \ldots \cdot \xi_p) = \sum_{i=0}^{p} \xi_i\{\lambda(\xi_0, \ldots, \hat{\xi}_i, \ldots, \xi_p)\} \\
&\qquad + \sum_{0<i<j<p} (-1)^{i+j} \lambda([\xi_i, \xi_j], \xi_0, \ldots, \hat{\xi}_i, \ldots, \hat{\xi}_j, \ldots, \xi_p) \\
&(L_\xi \lambda)(\xi_1, \ldots, \xi_p) = \xi\{\lambda(\xi_1, \ldots, \xi_p)\} \\
&\qquad - \sum_{i=1}^{p} \lambda(\xi_1, \ldots, [\xi, \xi_j], \ldots, \xi_p) \\
&\{i(\xi)\lambda\}(\xi_1, \ldots, \xi_p) = \lambda(\xi, \xi_1, \ldots, \xi_p)
\end{aligned}\right. \tag{119}$$

we obtain respectively an antiderivation d *of grade* 1, *a derivation* L_ξ *of grade* 0, *and an antiderivation of grade* -1 *of the graded commutative algebra* $\Lambda^*(M)$ (28), *such that, for* $\xi, \eta \in \mathscr{X}(M)$

---

(26) I.e. $\xi a = 0, a \in R (a \in C)$.

(27) An antiderivation of the Grassmann algebra is a graded derivation of odd grade.

(28) The properties (63), (64) where $\delta_0 = d$, $\Theta_0(\xi) = L_\xi$ are part of this statement. d is the *exterior derivative*, L_ξ the *Lie derivative*. The product (60) is, in this case the usual wedge product of Cartan forms.

(120) $$d^2 = 0$$

(121) $$i(\xi)^2 = 0$$

(121 bis) $$L_\xi L_\mu - L_\mu L_\xi = L_{[\xi,\mu]}$$

(122) $$d \cdot i(\xi) + i(\xi) \cdot d = L_\xi$$

(123) $$d \cdot L_\xi = L_\xi \cdot d = d \cdot i_\xi \cdot d$$

(124) $$i(\xi) \cdot L_\mu - L_\mu \cdot i(\xi) = i([\xi, \mu])$$

(125) $$i(\xi) \cdot (\Phi \wedge) + (\Phi \wedge) \cdot i(\xi) = \Phi(\xi)\mathbf{1} \quad ,\Phi \in \Lambda^1(M).$$

The corresponding cochain complex $\{\Lambda^*(M), d\}$ *is the* De Rham complex of M, *with cohomology* $H^*(M)$, *the* De Rham cohomology of M, *both algebras under the wedge product.*

Let now E *be a smooth vector field on* M *of rank* r, *with* $\Gamma(E)$ *the* $\Lambda^0(A)$-*module* (29) *of smooth sections of* E, *and let* D *be a linear connection of* E, *i.e., for each* $\xi \in \mathcal{X}(M)$, D_ξ *is a linear operator of* $\Gamma(E)$ *s.t.*

(126) $$\begin{cases} D_\xi(aX) = aD_\xi X + (\xi a)X & X \in \Gamma(E) \\ D_{a\xi} = aD_\xi & , \quad a \in \Lambda^0(M). \end{cases}$$

We denote by $\Lambda^p(M, E)$ *the set of* E-*valued smooth* p-*forms on* M (30), *and define, for* $\lambda \in \Lambda^p(M, E)$, *and* $\xi_0, \ldots, \xi_p \in \mathcal{X}(M)$

(127) $$\begin{cases} (D\lambda)(\xi_0, \ldots, \xi_p) = \sum_{i=0}^{p} (-1)^i D_{\xi_i}\{\lambda(\xi_0, \ldots, \hat{\xi}_i, \ldots, \xi_p)\} \\ \quad + \sum_{0 \leqslant i < j \leqslant p} (-1)^{i+j} \lambda([\xi_i, \xi_j], \xi_0, \ldots, \hat{\xi}_i, \ldots, \hat{\xi}_j, \ldots, \xi_p) \\ \{\Theta(\xi)\lambda\}(\xi_1, \ldots, \xi_p) = D_\xi \; \lambda(\xi_1, \ldots, \xi_p) \\ \quad - \sum_{i=1}^{p} \lambda(\xi_1, \ldots, [\xi, \xi_i], \ldots, \xi_p) \\ \{i(\xi)\lambda\}(\xi_1, \ldots, \xi_{p-1}) = \lambda(\xi, \xi_1, \ldots, \xi_p). \end{cases}$$

We thus obtain $\Lambda^0(M)$-*linear operators on* $\Lambda^*(M, E) = \overset{n+r}{\underset{p=1}{\oplus}} \Lambda^p(M, E)$ *of respective*

---

(29) One has $\Lambda^0(M) = C^\infty(M)$.

(30) I.e. $\Lambda^p(M, E) = \Lambda^p(M), \Gamma(E))$, the set of alternate, $\Lambda^0(M)$-linear, $\Gamma(E)$-valued p-forms on $\Lambda^0(M)$.

grades 1, 0 *and* − 1, *such that* [31]

$$(128)\qquad \begin{cases} D(a\lambda) = aD\lambda + a \wedge \lambda & \lambda \in \Lambda^*(M \cdot E) \\ \Theta(\xi)(a\lambda) = a\Theta(\xi)\lambda + (\xi a)\lambda, & a \in \Lambda^0(M) \\ i(\xi)(a\lambda) = a\, i(\xi)\lambda & \end{cases}$$

and for $\xi, \eta \in \mathscr{X}(M)$

$$(129)\qquad i(\xi)^2 = 0, \quad \Theta_\xi \Theta_\mu - \Theta_\mu \Theta_\xi = \Theta_{[\xi,\mu]} + \Omega(\xi, \mu)$$

$$(130)\qquad D \cdot i(\xi) + i(\xi) \cdot D = \Theta(\xi)$$

$$(131)\qquad i(\xi) \cdot \Theta(\eta) - \Theta(\eta) \cdot i(\xi) = i([\xi, \eta])$$

Furthermore, D^2 *is an endomorphism of the* $\Lambda^0(M)$*-module* $\Lambda^*(M, E)$,

$$(132)\qquad D^2(a\lambda) = aD^2\lambda, \qquad \lambda \in \Lambda^*(M, E), a \in \Lambda^0(M)$$

given by

$$(133)\qquad D^2\lambda = \Omega \wedge \lambda \qquad \lambda \in \Lambda^*(M, E)$$

where Ω *is the* curvature of D, *defined by*

$$(134)\qquad (D^2 a)(\xi, \eta) = \{ D_\xi D_\eta - D_\eta D_\xi - D_{[\xi,\eta]} \} a, \begin{cases} a \in \Lambda^0(M) \\ \xi, \eta \in \mathscr{X}(M) \end{cases}$$

and the wedge product $\Omega \wedge \lambda$ *is defined as in (56). Furthermore, one has for* $\xi, \eta \in \mathscr{X}(M)$

$$(135)\qquad \Theta(\xi) \cdot D - D \cdot \Theta(\xi) = \{ i(\xi)\, \Omega \} \wedge = i(\xi) \cdot (\Omega \wedge) - (\Omega \wedge) \cdot i(\xi)$$

with $\{ i(\xi)\Omega \} \wedge$ *defined as in (57), and furthermore*

$$(135a)\qquad \Theta(\xi)\, \Theta(\eta) - \Theta(\eta)\, \Theta(\xi) = \Theta([\xi, \eta]) + \Omega(\xi, \eta).$$

We end up by describing the important notions of direct sums and tensor product of connections.

[11] PROPOSITION. *Let* (L, A) *be a Cartan pair; and let* V_1 *and* V_2 *be left* A-

---

[31] $a \wedge \lambda$ is defined in (55).

-modules, with ρ_1 a V_1-connection and V_2 a V_2-connection. We denote by S and P the respective direct sum and tensor product left A-module.

(136) $$S = V_1 \oplus V_2$$

(137) $$P = V_1 \underset{A}{\otimes} V_2 .$$

If, we set, for $X_1 \in V_1, X_2 \in V_2, \xi \in L$

(138) $$(\rho_1 \oplus \rho_2)(\xi)(X_1 \oplus X_2) = \rho_1(\xi)\, X_1 + \rho_1(\xi)\, X_2$$

(139) $$(\rho_1 \otimes \rho_2)(\xi)(X_1 \otimes X_2) = \rho_1(\xi)X_1 \otimes X_2 + X_1 \otimes \rho_2(\xi)X_2$$

we obtain an S-connection $\rho_1 \oplus \rho_2$ and a P-connection $\rho_1 \otimes \rho_2$. Furthermore, if ρ_1 and ρ_2 are local, so are $\rho_1 \oplus \rho_2$ and $\rho_1 \otimes \rho_2$. And we have the curvatures

(140) $$\Omega_{\rho_1 \oplus \rho_2} = \Omega_{\rho_1} \oplus \Omega_{\rho_2}$$

(141) $$\Omega_{\rho_1 \otimes \rho_2} = \Omega_{\rho_1} \otimes 1_{V_2} + 1_{V_1} \otimes \Omega_{\rho_2}$$

hence, if ρ_1, and ρ_2 are flat, so are $\rho_1 \oplus \rho_2$ and $\rho_1 \otimes \rho_2$.

Proof. The definition of $\rho_1 \oplus \rho_2$ amounts to

(138a) $$(\rho_1 \oplus \rho_2(\xi) = \rho_1(\xi) \oplus \rho_2(\xi), \qquad \xi \in L,$$

from which the $\mathbb{R}$-linearity (resp. $\mathbb{C}$-linearity) of $\rho_1 \oplus \rho_2$ is obvious. On the other hand, we have, for $\xi \in L, a \in A, X_1 \in V_1, X_2 \in V_2$

(142) $$\begin{aligned} &(\rho_1 + \rho_2)(\xi)a(X_1 \oplus X_2) = \\ &\qquad = \rho_1(\xi)aX_1 \oplus \rho_2(\xi)aX_2 \\ &\qquad = [a\rho_1(\xi)X_1 + (\xi a)X_1] \oplus [a\rho_2(\xi)X_a + (a\xi)X_2] \\ &\qquad = a(\rho_1 + \rho_2)(\xi)(X_1 \oplus X_2) + (a\xi)(X_1 \oplus X_2). \end{aligned}$$

The fact that $\rho_1 \oplus \rho_2$ is local if this holds for ρ_1 and ρ_2, as well as relation (140), are obvious from (138).

We recall that P is defined as the quotient of the free abelian group of sums $\sum_{i=1}^{n} X_1^i \otimes X_2^i$, $X_1^i \in V_1$, $X_2^i \in V_2$, $n \in \mathbb{N}$ by the subgroups generated by elements

(143) $$\begin{cases} X_1 \otimes (X_2 + X_2') - X_1 \otimes X_2 - X_1 \otimes X_2' \\ (X_1 + X_1') \otimes X_2 - X_1 \otimes X_2 - X_1' \otimes X_2 \\ X_1 \otimes aX_2 - aX_1 \otimes X_2 \end{cases}$$

where $X_1, X_1' \in V_1$, X_2, $X_2' \in V_2$, $a \in A$, the left A-module structure being

REFERENCES

[1] H. CARTAN in *Colloque de Topologie*, (Espaces fibrés), Bruxelles 1950, pp. 15 - 27 and 57 - 71.
[2] E. NELSON, *Tensor Analysis*, Mathematical Notes, n. 5, Princeton University Press, Princeton 1967.
[3] W. GREUB, S. HALPERIN, R. VANSTONE, *Connections Curvature and Homology*, Pure and Applied Mathematics Vol. 47 Academic Press, New York 1972.

Manuscript received: May 3, 1985.

Paper presented by
A. Lichnerowicz